ISSN 2583-1739

Svādhyāya - International Journal of Transdisciplinary Research and Development (SIJTRD)

Vol 1 (1), May 2021, pp 72-80

Role of Algebra in Graph Theory

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ABSTRACT

One of the important branches of Mathematics is Graph theory. The graph problems are solved using algebraic methods known as algebraic graph theory. The study of algebraic graph theory is carried out in this article and a new concept of Algebraic and Geometric multiplicity of an undirected graph to find the largest matching is introduced and discussed.

Key Words:* *Graph Theory, Matching, Maximum Matching, Geometric Multiplicity, complete graph, dense graph.

AMS Classification Key: 05C, 05C70, 911368, 15A18

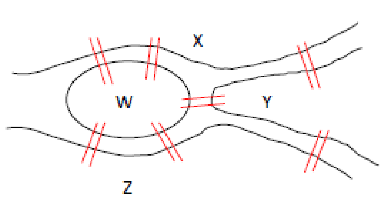
INTRODUCTION

One of the emerging branches of Mathematics is graph theory. In Graph theory the study of algebraic graph theory and its application is one of the interesting areas for researchers. Many research articles have been published and algebraic techniques are increasingly used in graph structures. In this study, the concept of algebraic graph theory and its applications are highlighted.

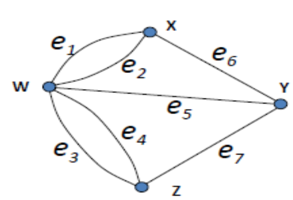
GRAPH THEORY

In mathematics, the study of mathematical structures in graph theory is to make pairwise relations between elements. A graph is formed by nodes (points) and it is connected by lines.

For example,



Konigsberg Bridge



Graph

The figure above represents the Konigsberg Bridge graph, $G_u = \{V, E\}$ where nodes V= {w, x, y, z} are the region on either side of the river and edges E = $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ are paths or bridges connecting the rivers.

ISSN 2583-1739

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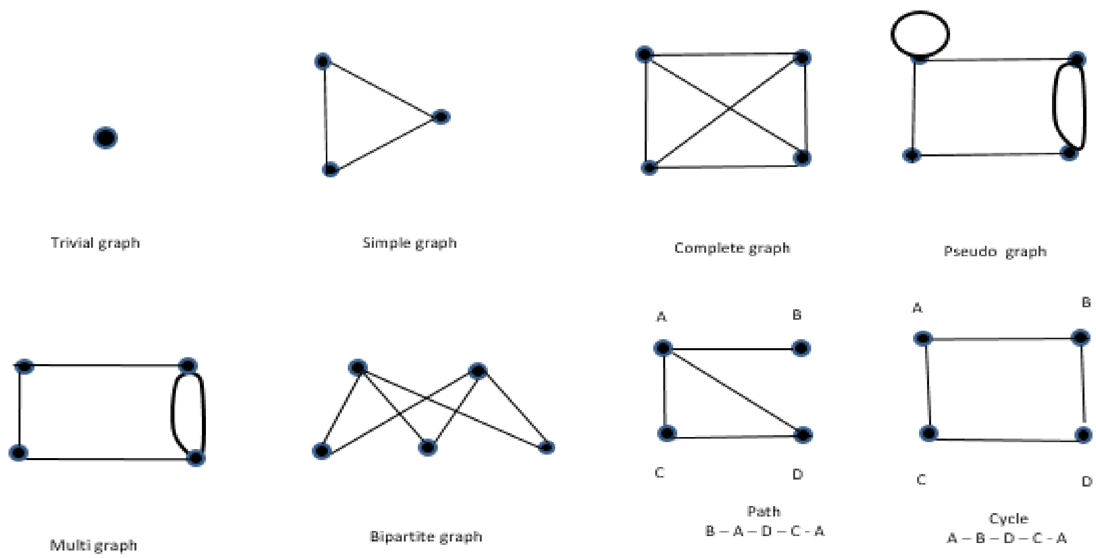
The concept of graphs in Graph theory consists of some basic terms such as Degree of vertices, Adjacency matrix, Incidence matrix, Path, Cycle, Walk , etc., and it also has some basic different types of graphs.

For example,



ALGEBRAIC GRAPH THEORY

Algebraic tools are used for elegant proofs in algebraic theory and there are many interesting algebraic objects associated with graphs. Now there are more books dealing with various aspects of this subject**.** The books written by N Biggs, 1993) and (Godsil and Royle, 2001 in graph theory based on certain algebraic concepts contains massive information. Here we study the research carried in algebraic graph theory mainly Linear and Abstract algebra.
Some basic definitions and example are explained before studying in detail about algebraic graph theory.

A square matrix n x n is said to be a symmetric if its elements $a_{ij} = a_{ji}$ or the transpose of matrix is the matrix itself. Also the symmetry property holds good in adjacency matrix of an undirected graph with or without self-loops.

The adjacency matrix is a (finite) square matrix whose components are expressed as 1 if the vertices pair are adjacent to each other and zero otherwise.
For example,

ISSN 2583-1739

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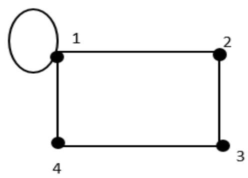
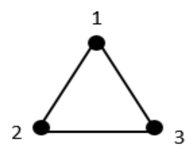


$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Undirected graph with self-loop

Symmetric and adjacency matrix

The spectrum of graph is the Eigen values of the adjacency matrix of a graph known as graph spectrum. For example, consider a graph with 3 points and 3 lines and it must be complete.



$$A= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

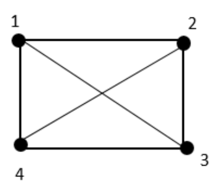
From the above adjacency matrix the characteristic equation is,

$$\lambda^3 - 3\lambda - 2 = 0$$

$\lambda = -1, -1, 2$ are the Eigen values of the graph and it's the spectrum of the graph.

In the adjacency matrix all elements except the diagonal element is formed by the degree of its vertices known as graph degree.

An undirected graph formed with no self-loops and multi edges is known as Laplacian matrix and it is given as L = D – A, where the degree of the matrix is represents D and the adjacency matrix of the graph is represented by A For example, draw an undirected graph with 4 vertices (points) and 6 edges (lines)



$$A= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad D= \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Adjacency matrix

Degree matrix

The Laplacian matrix is given by,

$$L=D-A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

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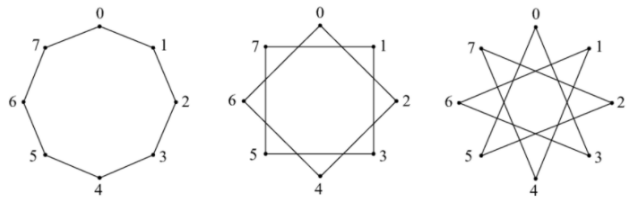
The circulant graph in a graph theory is an undirected cyclic group of graph symmetries which takes any vertex to any other vertex.

For example from (Rajasingh, 2013),



Algebra of graph is giving an algebraic structure to an undirected graph in graph theory and it has many uses in the field of universal algebra.

For example,

Consider graph G $_u$ whose vertices are positive integers and it is denoted by a pair *(v, e)* where *v* is the set of vertices and *e*, the set of edges is a subset of *v* x *v*. Then the algebraic structure is given by,

The algebraic structure $(G, +, \rightarrow, \varepsilon)$ formed in the above graph satisfies the usual laws,

- An empty set of the graph is ε
- Idempotent commutative monoid is$(G, +, \ \varepsilon)$
- Monoid is $(G, \times, \varepsilon)$
- × distributive over +, (e.g.) 1 × (2+3) = (1 × 2) + (1 × 3)

STUDY OF GRAPH THEORY USING LINEAR ALGEBRAIC CONCEPT

In modern presentations of geometry the fundamental is linear algebra including for defining basic objects like lines, planes and rotations.

The study in Algebraic graph theory started with spectrum of adjacency matrix or Laplacian matrix in Linear algebra known as spectral graph theory and it involves the graph connection with linear algebra. A diagonalizable matrix can be factorized to a canonical form which represents the matrix in terms of Eigen values and Eigen vectors are known as its spectrum. The Eigen values and Eigen vectors is studied in spectral graph theory. A real symmetric matrix is orthogonally diagonalizable and its Eigen values are real algebraic integers.

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In a finite simple graph the adjacency matrix is a (0-1) matrix with zeros on its diagonal and in an undirected graph the matrix is symmetric. The relation between a graph and the Eigen value, Eigen vectors of its adjacency matrix are studied in a spectral graph.

Linear Algebra Applied to graph theory (Paul M Nguyen, 2017) it an application to linear algebra using graph theory and it is a best method of studying traffic flow for its networking, electronic circuits or delivery routes. To represent a graph mathematically consider and establish various matrix operation which are well defined for adjacency matrix.

Graph theory and Linear algebra (Dylan, 2017) explains the relation between matrix representations and the matrix properties found in linear algebra. It identifies certain unique properties of special classes of graph such as complete graph and acyclic graph with their specialty in graph theory that reflects in matrix properties.

Graph polynomials are applied to characterize networks in Thermodynamics (Cheng Ye, 2015). This represents how the characteristic equation of the normalized Laplacian matrix of a graph structure is linked with Boltzmann partition function. For this network many thermodynamic quantities are found, in this average energy and entropy are also included. This finds an importance in real life networks involving financial and biological domains which are usually complex in general.

An algebraic graph representation to enumerate connected graphs is studied in (Mestre, 2013). In this paper a recursion formula is obtained to generate all equivalence classes of connected graph.

The combinatorial properties of the set of all m x n matrices of zeros and ones having r_i ones in row i and c_j ones in column j is studied for all positive integers m and n (Richard 1980) . A bipartite graphs with bipartition into m and n vertices having degree sequence R and S respectively is formulated using the results.

STUDY OF GRAPH THEORY USING ABSTRACT ALGEBRA:

The study of algebraic structure is known as Group theory. Rings, Fields are well-known algebraic structures with some additional operations and axioms.

The study of automorphism groups and geometric group theory find its space in the study of graph theory involving algebraic tools. Few examples are symmetric graphs, vertex transitive, edge transitive graphs, etc. are mostly involved in this study.

The symmetry form of graph which is mapped onto itself in which edge- vertex connectivity is preferred is known as Automorphism of a graph and this can be applied for both directed and undirected graphs.

The study of finitely generated groups creating connection between algebraic properties of such groups and geometric properties in which the group acts. Also finitely generated groups itself act as geometric objects is an important idea of geometric graph theory.

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An Automorphism group and a graph also deals with subgroup generation (Ivanov, 1994). The s-transitive group of the graph act on the set of paths of length s in the graph and it is proved for 1-transitive group where s will be the longest integer such that subgroup acts s in a transitive manner.

Finite groups also gets a way in graph study (Edward A Bertram, 1982) where the results concern about the degree sequence, vertex colorings and vertex independence number.

Group theory application is studied over molecular systems biology (Edward A Rietman, 2011). This provides a mathematical idea to study the boundaries of living and non-living systems.

ALGEBRAIC AND GEOMETRIC MULTIPLICITY OF UNDIRECTED GRAPH:

One of the most concept of Graph theory is matching theory. In this concept the largest cardinality of various types of graphs have been studied. Study of finding maximum matching of an undirected graph through algebraic theory based on its multiplicity of Eigen values is a new research concept in case of undirected graph whereas this idea is already analysed in directed graph (Yunyun, 2016).

Let A be an $n \times n$ square matrix with one Eigen value λ. In the characteristic polynomial the number of times λ repeated as a root of the characteristic polynomial is referred as algebraic multiplicity. For every eigen value corresponding eigen vectors can be found. The number of linearly independent eigenvectors for an eigen value gives the geometric multiplicity of the eigen value.

Algebraically the geometric multiplicity is nothing but the dimension of the null space.

It has been studied that the two multiplicities (algebraic and geometric) factors of eigen values of a matrix may be different but the geometric multiplicity will never exceed algebraic multiplicity. Also if the algebraic and geometric multiplicities equals for very eigen value of the matrix then the corresponding matrix is diagonalizable.

In the study of complex eigen values of population operator and algebraic multiplicity (Xue-zhi Li, 2001) it is proved that the complex Eigen value of any operator the algebraic multiplicity is atmost 1 and which inturn provides the solution of corresponding population system in an asymptotic expansion.

UNDIRECTED COMPLETE GRAPH AND ALGEBRAIC MULTIPLICITY

Consider the adjacency matrix A of order *m* of a complete graph with unit weight assuming no self-loops. Let

$$A = \begin{bmatrix} 0 & 1 \ldots & 1 \\ 1 & 0 \ldots & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & 1 \ldots & 0 \end{bmatrix}$$

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The characteristic polynomial of the matrix A is given by,

$$|\lambda I_m - A| = [\lambda - (m-1)](\lambda+1)^{m-1}$$

Then the eigen values and the corresponding algebraic multiplicities are given below,

if $eigen\ value\ \lambda_1 = m-1\ then\ corresponding\ algebraic\ multiplicity\ \delta(\lambda_1) = 1$ and if $eigen\ value\ \lambda_2 = -1\ then\ \delta(\lambda_2) = m-1$

$$\therefore \sum_{j=1}^{l} \delta(\lambda_j) = \delta(\lambda_1) + \delta(\lambda_2) = 1 + m - 1 = m$$

As the sum of algebraic multiplicities equals the order of the matrix we can conclude that all the nodes are matched nodes.

GEOMETRIC MULTIPLICITY OF AN UNDIRECTED SPARSE GRAPH AND DENSE GRAPH (YUAN, 2013)

For an undirected graph the greatest geometric multiplicity $\mu(\lambda_j)$ of the eigen value λ_j of A_j

$$\mu(\lambda_j) = dim\ dim\ V_{\lambda_j} = N - rank\ \{\lambda_j I_N - A\}$$

Where $\lambda_j (j = 1, 2, 3, \dots, N)$ represent the distinct eigen values of A and I_N is the unit matrix with the same as A.

In an undirected sparse graph if the rows of the adjacency matrix are linearly independent the Eigen value $\lambda_j = 0\ (j = 1, 2, 3, \dots, N)$. Then the Geometric Multiplicity $\mu(\lambda_j) = 0$. This gives that all the nodes of the graph is perfectly matched and the corresponding edges are perfectly matched. Since the maximum matching based on geometric multiplicity gives the number of unmatched nodes.

In an undirected dense graph, $det\ det\ (\lambda_m I_N - A) = 0$ if $\lambda = -1$ and it is the eigen value of Largest Geometric Multiplicity. Then column transformation is performed to the matrix $(\lambda_m I_N - A)$ to obtain column canonical and the linearly independent rows are combined with linearly dependent rows to form a new matrix which gives the matched nodes and the corresponding matched edges.

PERFECT MATCHING OF BRIDGE GRAPH BASED ON GEOMETRIC AND ALGEBRAIC MULTIPLICITY (JINI.J, 2020)

A cubic undirected bridge graph is perfectly matched if n_i- Bridges (i=1, 2, 3… n) are connected in a single path. For an undirected cubic graph with zero bridge the perfect matching is based on Algebraic multiplicity and for n_i - Bridges the perfect matching is based on Geometric multiplicity. In this paper an undirected cubic graph connected in a single path has been found up to 3 bridges and the concept is extended to n_i - Bridges.

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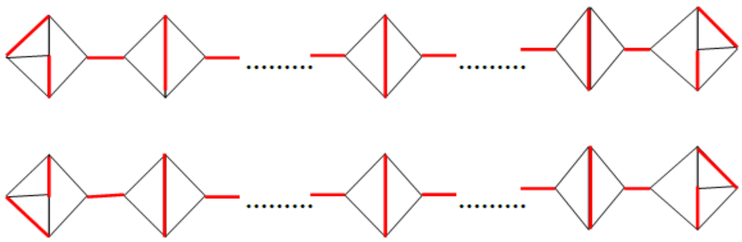

GEOMETRIC MULTIPLICITY AND TENSOR ANALYSIS

Geometric multiplicity of eigen values of tensors and the geometric and algebraic multiplicity of irreducible tensors eigen values is studied **(Li, 2015)** and the it is found that the eigen values with modulus have the same geometric multiplicity. Also it has been proved that in two-dimensional nonnegative tensors geometric multiplicity of eigenvalues is equal to algebraic multiplicity of eigenvalues.

CONCLUSION

In this paper, literature survey is made for the study of algebraic theory in Graph theory by the definitions and uses of first three branches of Algebraic theory. Also some research articles based on algebraic theory was studied to perform further research in the field of algebraic Graph theory. Finally my research in algebraic graph theory for finding maximum matching of an undirected graph based on algebraic and Geometric multiplicity is explained briefly.

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